PROBLEM SOLVING has been selected as the focus of mathematics instruction for the eighties by the National Council of Teachers of Mathematics. The revised California State Framework for Mathematics will emphasize the integration of problem solving into all of the other strands.

It is, therefore, essential that all teachers of mathematics gain a firm and clear understanding of the nature of a problem, the process of problem solving, an enthusiasm for problem solving that arises out of personal experience, and knowledge of how to teach problem solving to students.

The major contributor to problem solving in mathematics is, undoubtedly, George Polya. This writer had the privilege of studying under Dr. Polya for two years at Stanford University as well as engaging in many informal discussions with him both individually and in small groups. Frequent reference will be made to his work in this paper. The 1980 Yearbook of NCTM, entitled Problem Solving in School Mathematics relies heavily upon Polya's contributions. It is highly recommended as a source book.

Here are quotes from Dr. Polya:

> "Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking."
>
> - "How to Solve It", Polya

On solving mathematical problems, Dr. Polya writes:

1. "Solving a problem is finding the unknown means to a distinctly conceived end...To solve a problem is to find a way where no way is known off-hand...by appropriate means. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of man. The ability to go round an obstacle...raises man far above the most clever animals, and men of talent above their fellow men...The greater part of our conscious thinking is concerned with problems.
2. If education fails to contribute to the development of the intelligence, it is obviously incomplete. Yet intelligence is essentially the ability to solve problems...
3. In my opinion, the first duty of a teacher of mathematics is to use this great opportunity; he should do everything in his power to develop his students' ability to solve problems.

   First, he should set his students the right kind of problems; not too difficult and not too easy, natural and interesting, challenging their curiosity, proportionate to their knowledge. He should also allow himself some time for presenting the problem appropriately, so that it appears in the proper light.

Then, the teacher should help his students properly. Not too little, or else there is no progress. Not too much, or else the student has nothing to do...If the teacher helps his students just enough and unobtrusively...they may experience the tension and enjoy the triumph of discovery. Such experiences may contribute decisively to the mental development of the students.

4. Yet there is a first condition. Nobody can give away what he has not got. No teacher can impart to his students the experience of discovery if he has not got it himself. Therefore, in the opinion of the author, the curriculum for future teachers of mathematics should emphasize much more than it usually does nowadays, the practical ability to solve not too advanced problems and the methods of solution."

-California Mathematics Council Bulletin, Vol. 7 #2.

The position paper of the National Council of Supervisors of Mathematics (1977) contains this statement about problem solving in their list of ten basic mathematical skills:

"Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. The should be unfearful of arriving at tentative conclusions, and they must be willing to subject these conclusions to scrutiny."

The recent publication, "Problem Solving, a Handbook for Teachers" by Stephen Krulik and Jesse A Rudnick published by Allyn and Bacon, Inc. is highly recommended. The following outline is taken from that publication. Direct quotations are in quotation marks.

What is a problem?

"A problem is a situation, quantitative or otherwise, that confronts an individual or group of individuals, that requires resolution, and for which the individual sees no apparent or obvious means or path to obtaining the solution."

"Thus a problem must satisfy the following three criteria...

1. Acceptance: The individual accepts the problem. There is a personal involvement...
2. Blockage: The individual's initial attempts at solution are fruitless. His or her habitual responses and patterns of attack do not work.
3. Exploration: The personal involvement identified in (1) forces the individual to explore new methods of attack."

"While most mathematics textbooks contain sections labeled

"verbal problems", not all are really problems. In many cases, a model solution has already been presented in class by the teacher...We consider these...to be "exercises" or "routine problems."...However, a teacher should not think that students who have been solving these routine exercises through use of a carefully developed model or algorithm have been exposed to problem solving."

What is Problem Solving?

"Problem solving is a process. It is the means by which an individual uses previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. The student must sunthesize what he or she has learned, and apply it to the new...situation."

Why Teach Problem Solving?

"An emphasis on problem solving in the classroom can lessen the gap between what the real world and the classroom world and thus set a positive mood in the classroom...Problems show the interconnection among mathematical ideas...Problem solving is more exciting, more challenging, and more interesting to children than barren exercises...Problem solving permits students to learn and to practice heuristic thinking."

When Do We Teach Problem Solving?

"Experiences in problem solving are always at hand. All other activities are subservient. Thus the teaching of problem solving should occur in virtually every class period."

What Makes a Good Problem Solver?

"Problem solvers like to solve problems because they exist...are extremely persistent...a variety of methods of attack are usually at their disposal...show an ability to skip some of the steps in the solution process...are not afraid to make educated guesses... hold conversations with themselves."

What Makes a Good Problem?

1. The solution involves a distinct mathematical concept or skill.
2. The problem can be generalized or extended to a variety of situations.
3. The problem lends itself to a variety of solutions.

In the SMSG publication "Secondary School Mathematics" appears a set of objectives for teaching problem solving. They are:

1. to provide the student with a variety of strategies for problem solving;
2. to develop some flexibility in the student's approach to problem solving;
3. to develop techniques for using geometric representations as a way of developing new information about a given situation
4. to develop some skill in using a tabular arrangement of given and derived information; and
5. to develop a better understanding of a problem by teaching the student to make numerical estimates and testing them in the actual problem.

The following is taken from the 1980 Yearbook of NCTM, and strongly reflects Polya's contribution to Problem Solving.

How to Solve It

Understanding the Problem

1. What is the unknown? What are the data? What is the condition?
2. Is it possible to satisfy the condition?
3. Draw a figure. Introduce suitable notation.
4. Separate the various parts of the condition. Can you write them down?

Devising a Plan

1. Have you seen it before? Or have you seen it in a slightly different form?
2. Do you know a related problem?
3. Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.
4. Here is a problem related to yours and solved before. Could you use it?
5. Could you restate the problem?
6. If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem, a more general problem, a more special problem, or an analogous problem? Could you solve part of the problem?
7. Did you use all the data? Did you use the whole condition?

Carrying Out the Plan

Carrying out your plan of the solution, check each step. Can you prove it is correct?

Looking Back

1. Can you check the result? Can you check the argument?
2. Can you derive the result differently? Can you see it at a glance?
3. Can you use the result, or the method, for some other problem?

Structuralism
Wundt

"The ideas of which psychology seeks to investigate the attributes, are identical with thoseupon which natural science is based....the natural sciences....concern themselves with the objects of experience, thought of as independent of the subject" while the science of psychology was defined as an examination of "the whole content of experience in its relations to the subject and also in regard to the attributes which this content derives directly from the subject." W. Wundt, Outline of Psychology (7th edition 1907) p. 2-3.
Psychology studies "immediate experience" while natural science studies "mediate experience"; 'looking-in' vs 'looking-at' 'consciousness' (subject) studied by the method of 'introspection'.

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Behaviorism
Watson

"Psychology as the behaviorist views it is a purely objective experimental branch of natural science. Its theoretical goal is the prediction and control of behavior. Introspection forms no essential part of its methods, nor is the scientific value of its data dependent upon the readiness with which they lend themselves to interpretation in terms of consciousness. The behaviorist, in his efforts to get a unitary scheme of animal response, recognizes no dividing line between man and brute. The behavior of man, with all of its refinement and complexity, forms only a part of the behaviorist's total scheme of investigation."
J.B. Watson, 'Psychology as the Behaviorist Views It', Psychological Review 20 (1913) 158-177.

Dialectical Psychology
Riegel

"A dialectical theory of human development focuses on the simultaneous movements along at least the following four dimensions:
(1) inner-biological,
(2) individual-psychological,
(3) cultural-sociological, and
(4) outer-physical."
Klaus F. Riegel, 'The Dialects of Human Development' American Psychologists 31 (1976) 689-698.
Buss

"The way out of the viscious subject-object revolutionary circle is to effect a revolution to end revolutions. Such a revolution is within our grasp today. The new paradigm would embrace the idea that the subject-object relation is two-dimensional, rather than one-dimensional. Such a paradigm would transcend the limited perspectives of the various partialist paradigms. The paradigm to end structural revolutions involving the subject-object transformation is a dialectical one.
A dialectical paradigm emphasizes the reciprocal, interactive relationship between the person and reality such that each may serve as both subject and object."
Allan B. Buss, 'The Structure of Psychological Revolutions' Journal of the History of the Behavioral Sciences (1978) 57-64.

Humanistic Psychology

"In brief, a humanistic psychology emphasizes all of the following: the individual as a subject in the world rather than an object; the freedom and dignity of the individual rather than the individual as determined and dehumanized; the whole person rather than a fractionated being; and the persons' inherent potential for self-fulfillment, actualization, initiative action, and creativity rather than the individual as a merely passive respondent to his/her environment."

Allan R. Buss, 'Development of Dialectics and Development of Humanistic Psychology', Human Development 19 (1976) p. 252.

Psychoanalytic Psychology

"The intention of this project is to furnish us with a psychology which shall be a natural science: Its aim, that is, is to represent psychical processes as quantitatively determined states of specifiable material particles and so to make them plain and void of contradictions."

Sigmund Freud, Project for a Scientific Psychology (1895)

Some Questions to Guide Your Study

1. What appears to be the basic subject-object 'deep-structure' of the theologians you have read?

2. How does the assumption of a particular 'deep-structure' shape the methodology of a theologian?

3. What is the relationship between the assumption of a particular 'deep-structure' and the formulation of a specific doctrine, e.g., man?

4. What problems (empirical and conceptual) are solved by a specific doctrine based upon the assumption of a particular 'deep-structure'?

5. What problems (empirical and conceptual) are raised by the doctrine?

6. What type of problems seem to have been the most important or were judged to have the greatest 'weight'? Why should this have been the case?

7. Does there seem to have been a shift in theological 'paradigms' or 'traditions'? How would you characterize such a shift?

8. What factors (internal and external) might have influenced a 'theological revolution'?

9. Is it possible to speak of a theological doctrine or tradition as being more 'progressive' than another? By what criteria?

10. What would be necessary to translate or integrate a theological doctrine of one paradigm or tradition into the paradigm or tradition of a rival? Could this be done?