Skolem's method, with addition of the axioms of replacement, due to Fraenkel, and the axioms of excluded infinite regress due to John von Neumann (later introduced independently also by Zermelo).

21. Richard.—Jules Richard published in 1905 the antinomy now known by his name. Semantical paradoxes related to Richard's were afterward proposed by various other authors, including the paradox of Kurt Grelling in 1908 (*see* SEMANTICS IN LOGIC).

22. Hilbert.—David Hilbert's contributions to logic arose from his work in the foundations of mathematics (*see* MATHEMATICS, FOUNDATIONS OF). Important among them are the program of proof theory and in particular of a metatheoretic consistency proof, dating from 1905, and in connection with this the sharp distinction between object language and meta-language (in Hilbert's terminology, between mathematics and metamathematics). *Grundlagen der Mathematik* (1934, 1939), a comprehensive treatise of modern logic containing Hilbert's ideas in their final form, was written in collaboration with Paul Bernays, to whom the detailed content of the work is largely due.

23. Brouwer.—Luitzen Egbertus Jan Brouwer is the founder of mathematical intuitionism (*see* MATHEMATICS, FOUNDATIONS OF; and THOUGHT, LAWS OF). His publications in this field began with his dissertation in 1907 and a paper on the law of excluded middle in 1908 and extended through 1954. The logistic formalization of intuitionism is, however, due to Arend Heyting (1930) and others.

24. Lewis.—Clarence Irving Lewis was led by the "paradoxes" of material implication (*see* LOGIC) to seek a notion of implication, *strict implication,* which shall correspond rather to the relation of logical consequence in the sense that, if $<$ is the sign of strict implication, and if **A** and **B** are any sentences, $\mathbf{A} < \mathbf{B}$ shall be true if and only if **B** is a logical consequence of **A**. Lewis's publications about the matter begin in 1912. But the first satisfactory formulation of a propositional calculus with strict implication was in 1920. The book of Lewis and C. H. Langford, *Symbolic Logic* (1932), treating the subject at length, has become a classic in the field of modal logic, and the starting point of many more recent investigations.

25. Loewenheim.—Leopold Loewenheim in a paper of 1915 proved the theorem which is now known as Loewenheim's theorem (*see* LOGIC), and several other important results in the metatheory of the functional calculus of first order.

26. Skolem.—Thoralf Skolem, besides the contribution to set theory already mentioned, gave a new and better proof of Loewenheim's theorem, established the extension of this theorem which is stated in the article LOGIC, contributed results connected with the decision problem of the functional calculus of first order (some of which were later important in the proof of Goedel's completeness theorem), and discovered also the following metatheorem, that no set of postulates, finite in number or enumerably infinite, expressible in the notation of a simple applied functional calculus of first order can be adequate for arithmetic in the sense of characterizing completely the system of non-negative integers.

27. Post.—Emil L. Post's dissertation of 1920, published in 1921, contains the first comprehensive metatheoretic treatment of a logistic formalization of the two-valued propositional calculus, including proofs of consistency and completeness; also the first formulation of a many-valued propositional calculus from a point of view which is abstract in the sense of being concerned with the form of the calculus independently of any particular interpretation.

28. Lukasiewicz.—Jan Lukasiewicz, in a paper of 1920, introduced a three-valued propositional calculus based on Aristotle's doctrine of future contingents (*see* THOUGHT, LAWS OF). This was later generalized to an analogous n-valued propositional calculus, different from that of Post. Much important work is also due to Lukasiewicz in the two-valued propositional calculus, and in the history of logic.

29. Tarski.—Alfred Tarski contributed extensively to two-valued and many-valued propositional calculus, taking his departure from the work of Lukasiewicz. However, his most noteworthy contributions, beginning in 1930, are to the general metatheory of logistic systems, a domain in which many important new ideas are due to him. Especially semantics, in the sense of the metatheoretic treatment of notions related to those of meaning and truth, is the creation of Tarski (*see* SEMANTICS IN LOGIC). Much of the more recent work of Tarski has been in the boundary region between logic and mathematics, or has applied methods and results of modern logic to special branches of mathematics.

30. Carnap.—Rudolf Carnap, in his *Der logische Aufbau der Welt* (1928), *Testability and Meaning* (1936–37) and many other publications, was a pioneer in the systematic application of the methods of modern logic in epistemology and philosophy of science—making in this a contribution to philosophic method which in the eyes of many exceeds in importance his support of a particular philosophical outlook (that of logical positivism, *q.v.*). Carnap's contributions to the study of the metatheory of logistic systems begin in his *Logische Syntax der Sprache* (1934, published, 1937, in English with some additions as *The Logical Syntax of Language*). In a paper of 1935, somewhat later than Tarski but independently (and in a different terminology), Carnap introduced the idea of syntactical definitions of the semantical notions of truth and satisfaction, and in particular was the first to make such definitions for the full simple theory of types, as distinguished from a functional calculus of finite order. And concerning Carnap's contributions to intensional semantics, *see* SEMANTICS IN LOGIC.

31. Herbrand.—Jacques Herbrand in his short life—he was killed in a mountain-climbing accident in 1931 at the age of 23—made extensive contributions to Hilbertian proof theory and to the metatheory of the functional calculus of first order. The most important of these cannot be stated here. But the deduction theorem of first-order functional calculus should be mentioned as Herbrand's.

32. Goedel.—Kurt Goedel proved the completeness theorem of the pure functional calculus of first order (1930), and the famous incompleteness theorem (1931). For these, *see* LOGIC and (especially for the latter) MATHEMATICS, FOUNDATIONS OF.

The bearing of the incompleteness theorem on Hilbert's program of a metatheoretic consistency proof for mathematics is obvious; but more far-reaching is the consequence that no single logistic system, satisfying certain very general conditions, can tenably claim to embrace only logical truth *and* the whole of logical truth (if indeed the latter phrase has a meaning at all).

Also due to Goedel (1940) is the metatheorem that if the system of set theory with omission of the axiom of choice (*see* LOGIC) is consistent, it remains so upon addition of the axiom of choice or an axiom expressing the generalized continuum hypothesis or both. (For a statement of the continuum hypothesis *see* MATHEMATICS, FOUNDATIONS OF.) As Goedel pointed out, his result is applicable alike to various forms of set theory and to type theory. The solution of the continuum problem, as the problem concerning the continuum hypothesis is known, was completed in 1964 by Paul J. Cohen, who showed on the same assumption of consistency that the axiom of choice is independent of the other axioms of set theory, and the continuum hypothesis is then independent of the axioms of set theory including the axiom of choice.

See also references under "Logic, History of" in the Index.

(Ao. C.)

BIBLIOGRAPHY.—*Ancient Logic:* H. Scholz, *Geschichte der Logik* (1931); I. M. Bocheński, *Ancient Formal Logic* (1951), *La Logique de Théophraste* (1947); J. Lukasiewicz, *Aristotle's Syllogistic From the Standpoint of Modern Formal Logic* (1951); Aristotle's *Prior and Posterior Analytics,* rev. text with introduction and commentary by Sir W. D. Ross (1949); B. Mates, *Stoic Logic* (1953).

Indian Logic:—General Introductions: S. Kuppusvami Sastri, *Essentials of Indian Logic* (1931); Visvanatha Nyayapancanana Bhattacarya, *Bhāṣā-pariccheda With Siddhāntā-muktavālī,* trans. by Swami Madhavananda (1940); Alfred Foucher, *Le Compendium des Topiques Tarkasamgraha d'Annambhaṭṭa* (1949).

The Old Nyaya: Ganganatha Jha, *The Nyāyasūtras of Gautama With the Bhāṣya of Vātsyayana and the Vārtika of Uddyotakara. . .,* Eng. trans., published serially in *Indian Thought,* vol. 4–11 (1912–19), reprinted in book form with omission of Vartika of Uddyotakara (1939); H. N. Randle, *Indian Logic in the Early Schools* (1930); Giuseppe Tucci, *Pre-Dinnaga Buddhist Texts on Logic From Chinese Sources* (1929); T. (Fedor Ippolitovich) Stcherbatsky, *Buddhist Logic,* 2 vol. (1930–32); Stanislaus Schayer, "Studien zur indischen Logik," *Bulletin International de l'Académie Polonaise des Sciences et des Let-*

7

; the first individual variable (in alphabetic order) which is ifferent from both x and y.

To provide for the notion of an ordered pair, a notation $\{x, y,$ may be introduced to express that z is the ordered pair of x ıd y: $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} \rightarrow \mathbf{u}\epsilon\mathbf{z} \equiv_{\mathbf{u}} . [\mathbf{v}\epsilon\mathbf{u} \equiv_{\mathbf{v}} \mathbf{v} = \mathbf{x}] \vee . \mathbf{v}\epsilon\mathbf{u} \equiv_{\mathbf{v}} . \mathbf{v} =$ v y, where u and v are to be chosen as the first two individu riables different from x, y, and z. Relations may then dealt with in the theory by understanding a relation to be a t of ordered pairs.

The axioms of the theory are those of the functional calculus first order, and in addition the following axioms and axiom hemata:

Axiom of extensionality: $z\epsilon x \supset . z\epsilon x \equiv_z z\epsilon y \supset x = y.$
Axiom of the empty set: $\sim x\epsilon\Lambda.$
Axiom of pairing: $(\exists t) . z\epsilon t \equiv_z . z = x \vee z = y.$
Axiom of summation of sets: $(\exists t) . z\epsilon t \equiv_z (\exists y) . z\epsilon y \cdot y\epsilon x.$
Axiom of the set of subsets: $(\exists t) . z\epsilon t \equiv_z z \subset x.$
Axioms of subset formation: $(\exists t) . z\epsilon t \equiv_z . z\epsilon x \cdot \mathrm{A}z$, where ; is a well-formed formula which may have z as a free variable t does not have t as a free variable.
Axiom of choice: $y\epsilon x \cdot z\epsilon x \supset_{yz} [u\epsilon y \cdot u\epsilon z \supset_u y = z] \supset (\exists t) .$ $\supset_y . w\epsilon y \supset_w (\exists v) . u\epsilon y \cdot u\epsilon t \equiv_u u = v.$
Axiom of infinity: $(\exists t) . \Lambda\epsilon t . x\epsilon t \supset_x (\exists y) . y\epsilon t . z\epsilon y \equiv_z .$ $\vee z = x.$
Axioms of replacement: $y\epsilon x \supset_y (\exists u)[\mathrm{A}yz \equiv_z z = u] \supset (\exists t)$ $\epsilon t \equiv_z (\exists y) . y\epsilon x \cdot \mathrm{A}yz$, where $\mathrm{A}yz$ is a well-formed formula ıich may have y and z as free variables but does not have t or u a free variable.
Axioms of excluded infinite regress: $(\exists x)\mathrm{A}x \supset (\exists x) . \mathrm{A}x .$ $\supset_y \sim\mathrm{A}y$, where $\mathrm{A}x$ is a well-formed formula which may ve x as a free variable but does not have y as a free variable.

The way in which the theory seeks to avoid antinomy may be n in particular in connection with the axioms of subset forman. An uncritical formulation might well have included the onger axiom schema $(\exists t) . z\epsilon t \equiv_z \mathrm{A}z$, providing for the exence of a set t of those things z (sets and others) which satisfy arb ary given condition $\mathrm{A}z$. But this axiom schema would d tly to the Russell antinomy, upon taking $\mathrm{A}z$ to be ϵz. The weaker schema actually used (the axioms of subset mation) provides only for the existence of a set t of those ngs z which belong to a previously given set x and satisfy the ıdition $\mathrm{A}z$. This is not known to lead to antinomy; but $\sim(\exists t)$ $t \equiv_z \sim z\epsilon z$ is a theorem.

Concerning the history of these axioms for set theory, *see* the agraph about Zermelo in the article LOGIC, HISTORY OF. The oms of replacement and of excluded infinite regress are additions the original axiom system of Zermelo, and are sometimes itted, as although they are independent and for some purposes portant, there are also many purposes for which they are not ded.

On the other hand, if the axioms of replacement are retained the form in which we have here stated them, they have the ct of rendering the axioms of subset formation non-independ-; *i.e.*, the axioms of subset formation may then be omitted from list on the ground that they can be proved as theorems by ıg the axioms of replacement.

ın extension of the Zermelo set theory which is due to John von ımann and Paul Bernays (*see* the paper of Bernays cited in the iography) adds, to the sets of the Zermelo theory, also classes ch are not sets, *i.e.*, which cannot be members of other classes ets. And for every condition $\mathrm{A}z$ expressed in the notation of Zermelo theory there is a class of those elements z which sfy $\mathrm{A}z$—where an *element* is a set or anything capable of being ember of a set. (In particular there is, according to this exled theory, a class of elements z such that $\sim z\epsilon z$, but not a set uch elements.)

Two other, different, systems of set theory are treated by W. V. ne, one in his paper cited in the bibliography, and the other in 1 lition of his book. The latter, due jointly to Quine and Wang, is related to Quine's system of 1937. These are not ns of the Zermelo theory but seek to exclude the antinomies lifferent means.

Goedel's Incompleteness Theorem.—In contrast with the completeness theorem for the pure functional calculus of first order, all known systems of type theory or set theory which are of sufficient strength to provide a logical foundation of mathematics are incomplete. We state this incompleteness theorem, due to Goedel, in the slightly stronger form which was given to it by Barkley Rosser. On the hypothesis that the system in question—it may be in particular either the simple theory of types or any of the systems of set theory mentioned above—is consistent, there is a well-formed formula **A** which is a sentence (hence without free variables) such that neither **A** nor $\sim$**A** is a theorem.

The proof of the metatheorem proceeds by constructing a particular well-formed formula **A**, and then showing that, if the system is consistent, **A** cannot be a theorem, and $\sim$**A** cannot be a theorem. However, it can also be shown that, if the system is consistent, the proposition expressed by **A** is true. More accurately, we can show, by means that are formalizable in the system itself, "If the system is consistent, then ______," filling the blank with a statement of the proposition that is expressed by **A**. Hence the further metatheorem follows that the consistency of the system cannot be proved by means that are formalizable in the system itself. (This brief statement of the matter is incomplete, and some essential explanations which are here omitted will be found in the article MATHEMATICS, FOUNDATIONS OF.)

you can have only 2

1 complete
2 Consistent
3. power

Both the incompleteness theorem and the further theorem about the possibility of a consistency proof have the striking feature that they hold not only for a particular system but for any arbitrary extension of it that may be obtained by adjoining additional axioms or rules of inference or both, provided only that the requirement is satisfied that the axioms and rules of inference shall be effective (as described in the first part of this article, in explaining the notion of a logistic system). Having a particular system, which we suppose consistent, and having constructed the sentence **A** which is then true but not a theorem, we may indeed strengthen the system by adding **A** to it as an axiom. But the resulting stronger system is still incomplete: in it a particular sentence **B** may again be constructed by the same method, and it may be shown on the hypothesis of consistency that neither **B** nor $\sim$**B** is a theorem, and that the proposition expressed by **B** is true.

See also references under "Logic" in the Index.

BIBLIOGRAPHY.—Alfred Tarski, *Introduction to Logic,* 2nd ed. (1946); S. C. Kleene, *Introduction to Metamathematics* (1952); Alonzo Church, *Introduction to Mathematical Logic* (1956); D. Hilbert and P. Bernays, *Grundlagen der Mathematik,* 2 vol. (1934, 1939); A. A. Fraenkel, *Abstract Set Theory,* 2nd ed. (1961); Paul Bernays, "A System of Axiomatic Set Theory," *Journal of Symbolic Logic,* vol. 2–19 (1937–54); Hao Wang and Robert McNaughton, *Les Systèmes axiomatiques de la théorie des ensembles* (1953); W. V. Quine, "Element and Number," *Journal of Symbolic Logic,* vol. 6 (1941), *Mathematical Logic,* rev. ed. (1951); Barkley Rosser, "An Informal Exposition of Proofs of Gödel's Theorems and Church's Theorem," *Journal of Symbolic Logic,* vol. 4 (1939), *Logic for Mathematicians* (1953); C. I. Lewis and C. H. Langford, *Symbolic Logic* (1932; reissued 1951); Ruth C. Barcan, papers on modal logic in *Journal of Symbolic Logic,* vol. 11 (1946) and vol. 12 (1947); Rudolf Carnap, "Modalities and Quantification," *Journal of Symbolic Logic,* vol. 11 (1946), *Introduction to Symbolic Logic and its Applications* (1958); Kurt Schütte, *Beweistheorie* (1960). *See* also LOGIC, HISTORY OF and bibliographies given there. (Ao. C.)

LOGIC, HISTORY OF.

In this article the history of logic is presented as follows:

cision procedure would mean, so to speak, that the infinitely many arithmetical problems involve only a finite number of difficulties such that once those are overcome no further ingenuity will be required. But it might have seemed equally farfetched in 1918 to hope that mathematics could get a hold on the problem of finding a decision procedure for arithmetic, which would enable this problem to be solved by showing that there cannot be one. Exactly this latter was done in 1936 by Alonzo Church.

We begin by considering the computation problems for number-theoretic functions. What functions can be computed or effectively calculated, in the sense that, for a given argument, the value can be found by use only of preassigned rules, applicable without ingenuity on the part of the computer? Instead of a human computer, we can propose to have a machine to do the computation, according to rules built into its structure.

As this description is somewhat vague, what is needed is an exact mathematical formulation which comes to the same. Such a formulation was given briefly by Post in 1936 and independently in detail by Alan M. Turing (1936–37). What Turing did was to describe a kind of theoretical computing machine, not limited in use by a fixed maximum amount of storage of information nor liable to malfunctioning, as are actual computing machines. A Turing machine operates in the following manner. At discrete moments the machine assumes one of a finite list of states (fixed for a given machine), the first of which we call the passive state, the other k ($\geqq 1$) active. A linear tape, potentially infinite to the right, is ruled in squares, each of which is either blank or has printed upon it one of a finite number j ($\geqq 1$), of symbols (fixed for a given machine), including say the tally |; but only a finite number of squares are printed. The tape passes through the machine so that one of its squares is scanned. If at the given moment the machine is in an active state, then it can alter the condition of the scanned square (by erasing, printing, or both), and/or move the tape so that the scanned square becomes the one next right or left, and/or change to another state, between the given moment and the next; this act is determined (for a given Turing machine) by the state of the machine and the condition the scanned square at the given moment.

Turing used his machines primarily to compute in succession the digits of decimal expansions of real numbers, but they can be adapted to compute number-theoretic functions. Let an argument x be supplied to the machine by placing a blank followed by $x + 1$ tallies on successive squares of the tape starting with the leftmost square, leaving the tape otherwise blank, and placing the tape in the machine with the rightmost tally scanned and the machine in its first active state. The machine is said to compute a value y for x as argument, if then it eventually reaches the passive state with the $x + 1$ tallies followed immediately to the right by a blank and $y + 1$ tallies, the rightmost being scanned and the tape being otherwise blank. If, for each natural number x, the machine computes the correct value $y = f(x)$ of a function f, when supplied the argument x, it computes the function f, which is then said to be Turing computable.

Turing advocated the thesis that any function which can be effectively calculated is Turing computable. His arguments include showing that various operations that a human computer can perform can be analyzed into successions of the simple operations his machines perform. Church had already in 1936 proposed the thesis that every effectively calculable function is λ-definable in a sense due to Church and Stephen C. Kleene, or equivalently general recursive in a sense due to Goedel, who built on a suggestion of Jacques Herbrand; and this thesis was subsequently proved equivalent to Turing's, so the λ-definable functions, the general recursive functions, and the Turing computable functions are the same.

Church's Theorem.—The pattern of behaviour of a Turing machine is completely described by a $k \times (j + 1)$ table showing the act to be performed for each of the k active states with each of the $j + 1$ conditions of the scanned square. A system for encoding such tables can be established which, to each table, gives a natural number t as code number so that a person who knows the system can from t find the table and thence the behaviour of a machine M_t with this table. If t is the code number of a machine M_t which computes a value for x as argument, let $f_t(x)$ be this value; and let $f_t(x)$ be undefined otherwise.

Consider the function f defined by letting $f(x) = f_x(x) + 1$ if $f_x(x)$ is defined, and $f(x) = 0$ otherwise. This function f is not computable. For if machine M_q with code number q computes a value for q as argument, that value is $f_q(q)$, not $f_q(q) + 1$ as it would have to be if M_q computes f. It follows that the property "$f_x(x)$ is defined" is undecidable. For otherwise we could compute $f(x)$, given x, thus: first decide whether $f_x(x)$ is defined; then if it is, imitate the behaviour of M_x to compute it and add 1; if it is not, write 0.

Thus from Turing's thesis we have proved that there is a class of quite elementary arithmetical questions for which there is no decision procedure. This essentially is Church's theorem (1936), which he proved from his thesis using a corresponding class of questions in terms of λ-definability. For each x, the proposition "$f_x(x)$ is defined" is expressible in any formal system S which includes the usual vocabulary of arithmetic, by a formula A_x effectively determinable given x. If the quoted proposition is true, it is demonstrable intuitively by exhibiting the sequence of the acts of machine M_x in computing $f_x(x)$. Suppose S includes ordinary arithmetical reasoning; then that demonstration can be translated into a proof in S of A_x; *i.e.*, A_x is provable in S whenever (the proposition expressed by it is) true. Finally suppose about S that A_x is provable in S only when true. Then there can be no decision procedure for provability in S; for if there were, by answering the question whether A_x is provable in S, we would have a decision procedure for "$f_x(x)$ is defined."

Moreover there is no decision procedure for the important logical system F of first-order functional (or predicate) calculus, as Church (1936) and Turing (1936–37) showed. For a proof of A_x in S can be arranged as a deduction by F of A_x from a suitable finite set of arithmetical axioms independent of x, the conjunction of which can be written as one formula C. If the symbolism has been suitably chosen, then A_x is provable in S, if and only if the implication $C \supset A_x$ is provable in F. From these beginnings, a considerable theory has grown about "unsolvable decision problems," with further contributions by Post, A. A. Markov (the younger), Turing, Alfred Tarski, Andrzej Mostowski, P. S. Novikov, etc.

Goedel's Theorem.—A formal system like S is consistent, if for no formula B are both B and its negation $\sim$B provable in S; complete, if for each formula B without free variables (otherwise $\sim$B may not express the opposite of B) either B or $\sim$B is provable in S.

Now suppose further that $\sim A_x$ is provable in S only when true. Then it is absurd that S be complete (and that $\sim A_x$ be provable in S whenever true). For A_x is provable in S (when and) only when true. So if, for each x, either A_x or $\sim A_x$ were provable, we could decide whether $f_x(x)$ is defined or not, by searching through all the proofs in S till we find one of A_x or of $\sim A_x$. This is Goedel's theorem (1931) in a negative form.

To obtain the positive form, consider the process, for a given x, of searching through all the proofs in S for one of $\sim A_x$, and writing 0 if one is found. A Turing machine, say machine M_q with code number q, carries out this process, when supplied with x. Now by the choice of the formulas A_x, $\sim A_q$ is true if and only if $f_q(q)$ is undefined, which by the choice of M_q is the case if and only if $\sim A_q$ is unprovable in S. In brief, $\sim A_q$ expresses its own unprovability (*cf.* the paradox of the liar). So were $\sim A_q$ provable, it would be false, contradicting our supposition that $\sim A_x$ is provable only when true. So $\sim A_q$ is unprovable but true, and A_q is also unprovable (since false).

Hilbert's program called for embodying classical mathematics, including arithmetic, analysis, and set theory short of the paradoxes, in a formal system, and proving that system consistent by finitary methods. The part of Goedel's theorem that $\sim A_q$ is unprovable but true shows that not even the first step can be carried out fully. Despite the great value of the logistic method as a way of defining exactly the presuppositions of a given portion of mathematics, this very process of definition restricts that portion to include less of arithmetic than free use of intuitive reasoning would give, at least if what is already incorporated in the formal system is intuitively correct.

The part of the theorem that neither $\sim A_q$ nor A_q is provable in S shows the inadequacy of S to decide by proof or disproof, irrespective of the interpretation, all the statements expressible in it. Thus A_q is said to be formally undecidable in S.

In showing $\sim A_q$ unprovable (and true), we used (for $x = q$) the assumption that $\sim A_x$ is provable in S only when true. This can be replaced by the metamathematical assumption (not ostensibly using the interpretation) that S is consistent. For if $\sim A_q$ were provable, $f_q(q)$ would be defined, and hence as noted above A_q would be provable, which would contradict the consistency.

(The assumption that A_x is provable only when true, used in showing A_q unprovable, can be replaced by the assumption that S is *ω-consistent*, in the sense that for no formula $B(x)$ are $B(0)$, $B(1)$, $B(2)$, . . . and $\sim(x)B(x)$ all provable in S, where "(x)" expresses "for all x." In verifying this, $B(x)$ will express "at the x-th moment M_q has not yet computed $f_q(q)$." Adjoining A_q to S as a new axiom produces a formal system which is consistent but not ω-consistent, if S is consistent. Barkley Rosser in 1936 modified Goedel's formally undecidable formula so that consistency sufficed; in fact, letting M_r carry out the process of searching through all the proofs in S for one of $\sim A_x$ and writing 0 if one is found before one of A_x is found, $\sim A_r$ is unprovable but true and A_r is unprovable, if S is consistent.)

As Goedel observed in proving his theorem, the objects employed in a formal system, say S, are countable; and after correlating natural numbers (called now Goedel numbers) effectively to them, by talking about those numbers instead of the formal objects, the metamathematics of S becomes a part of arithmetic and so expressible in S. Now consider the implication "If S is consistent, then $\sim A_q$ is true," obtained above as part of Goedel's theorem. Let Consis be the formula of S which expresses "S is consistent" in S via the Goedel numbering. Then the quoted implication translated into S becomes Consis $\supset \sim A_q$. Goedel claimed that, by imitating formally in S the intuitive proof of his theorem, it should be possible to prove this formula in S. (This was done in detail by Hilbert and Bernays in 1939 in their formal system of arithmetic.) But then, if Consis were also provable, $\sim A_q$ would be, which is contrary to Goedel's (first) theorem if S is consistent. Thus Goedel obtained his second theorem: If S is consistent, then the formula Consis, which says in S via the Goedel numbering that S is consistent, is unprovable in S.

Goedel's second theorem raises an obstacle to the second part of Hilbert's program. One might have supposed that the finitary methods to be used in proving consistency would be part of the methods formalized in the system, which amounts to saying that a finitary proof of consistency of S should be formalizable as a proof in S of Consis. Only such methods appear to have been used in attempts prior to 1931. Goedel's second theorem shows that not even all the methods incorporated in S, including the nonfinitary ones, would suffice.

Some mathematicians felt that this ended once and for all Hilbert's plan to make classical mathematics more secure by a consistency proof. Others thought it possible that methods might be found for proving consistency, which, though not formalizable within S, could be construed as finitary and thus as safer than some of the methods incorporated in S. Thus Gerhard Gentzen in 1936 gave a proof of the consistency of a system S of classical arithmetic, in which the nonelementary method (not formalizable in S) is an extension of mathematical induction (*q.v.*) from the natural number sequence to a certain segment of the transfinite ordinal numbers which Cantor had introduced for extending the counting process beyond that provided by the natural numbers.

Further Perspectives.—Goedel's (first) theorem provided confirmation for the view advanced earlier by the intuitionists on philosophical grounds that the possibilities for intuitive mathematical thinking cannot be circumscribed in advance. Taking Brouwer's interpretation of a statement "a y exists such that $P(y)$" as meaning such a y can be found, and applying Church's thesis to the case the y depends on x, Kleene in 1943 proposed the further thesis that a statement "for each x, a y exists such that $P(x,y)$" is provable intuitionistically only if such a y can be given as a computable function of x. Using this thesis, there are theorems of classical arithmetic (Kleene, 1943) and of classical analysis (Ernst Specker, 1949) which are unprovable intuitionistically (not merely in some formal system, but absolutely); so classical mathematics reinforced by a consistency proof cannot serve as a tool of intuitionistic proof, except of a very restricted class of statements. Kleene (1945) and David Nelson (1946) together showed that the intuitionistic formal arithmetic based on Heyting's intuitionistic formal logic does conform to the thesis. Goedel in 1932–33 (partially anticipated by Andrei N. Kolmogorov in 1924–25) correlated to each arithmetical formula **B** a classically equivalent formula **B′** which is provable in the intuitionistic formal arithmetic if (and of course only if) **B** is provable in the classical formal arithmetic. In particular, the correlation proves very simply that the classical formal arithmetic is consistent if the intuitionistic is. However, the intuitionistic arithmetic distinguishes differing degrees of indirectness among statements classically equivalent.

Using the classical law of the excluded middle, Goedel in 1930 proved a completeness theorem for the first-order functional calculus F. In one form, this says that if a formal system obtained by adjoining mathematical axioms to F is consistent, it has a model; *i.e.*, there is an interpretation which makes the axioms true. Indeed, the model can be constructed using the natural numbers as the objects. In another form, the theorem says that in such a system every formula is provable which is true under all the interpretations of the undefined terms which make the axioms true. In view of this, when S is based on first-order functional calculus, the unprovability in S of $\sim A_q$ implies that, while $\sim A_q$ is true under the intended interpretation of the undefined terms, there is some other interpretation which makes the axioms true (a nonstandard model) under which $\sim A_q$ is false. This illustrates the theorem of Skolem (1933) that no list of axioms in the symbolism of the first-order functional calculus can characterize the natural numbers categorically. If higher functional calculus is used, the deductive apparatus will be incomplete. It thus appears that the logistic method is inadequate to characterize the natural numbers categorically. Peano's axioms characterize them categorically, but only through an interpretation which cannot be rendered fully through the deductive possibilities. Also we see the Goedel formal undecidability of A_q as a phenomenon of the same kind as the undecidability of Euclid's parallel postulate from the other postulates of Euclidean geometry.

Hierarchies of Properties.—By Church's thesis, the decidable properties of a natural number are countable; for, each decidable property P is decided by a Turing machine which computes 0 or 1 for x as argument according as x has the property P or not, and each Turing machine's behaviour is described by a code number. Thence the existence of undecidable properties follows simply by Cantor's theorem, by which the sets of natural numbers are uncountable; for, each property P corresponds to the set S of the natural numbers which have the property.

Church's theorem, however, goes beyond this: it exhibits an undecidable property of a very simple form. In our treatment, the undecidable property is "$f_x(x)$ is defined." Let "$T(t,x,y)$" abbreviate "t is the code number of a Turing machine M_t which, when supplied with x as argument, completes the computation of a value at moment y (but not earlier)," and let "(Ey)" abbreviate "there exists a y such that" or briefly "exists y" or "for some y." Then $[f_x(x)$ is defined$] \equiv (Ey)T(x,x,y)$ (for all x). But $T(t,x,y)$, and hence $T(x,x,y)$, is a decidable relation. So the undecidable property is of the form $(Ey)R(x,y)$ where $R(x,y)$ is a decidable relation. Using $\sim(Ey)T(x,x,y) \equiv (y)\sim T(x,x,y)$, there is also an undecidable property of the form $(y)R(x,y)$ with $R(x,y)$ decidable [where "(y)" means "for all y"]. Any decidable property $R(x)$ is expressible in both the forms $(Ey)R(x,y)$ and $(y)R(x,y)$, by taking $R(x,y) \equiv R(x)$ & $y=y$. But neither $(Ey)T(x,x,y)$ nor $(y)\sim T(x,x,y)$ is expressible in the form the other has. For, given any decidable relation $R(x,y)$, $(Ey)R(x,y) \equiv (Ey)T(p,x,y)$ where p is the code number of a Turing machine M_p which, when supplied with x as argument, searches for the least y such that $R(x,y)$. So, if $(y)\sim T(x,x,y) \equiv (Ey)R(x,y)$ held for some decidable R, we would have $(y)\sim T(p,p,y) \equiv (Ey)R(p,y) \equiv (Ey)T(p,p,y) \equiv \sim(y)\sim T(p,p,y)$, which is absurd. By similar